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Timelike-spacelike Mannheım partner curves in IR^3_1

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Abstract

In this paper, timelike-spacelike Mannheim partner curve couple was defined in Lorentzian space IR_1^3 and the relations were given between the curvatures and torsions of these curves. Furthermore, for a given curve couple, the necessary and sufficient conditions were obtained to become timelike-spacelike Mannheim partner curve couple in IR_1^3 .

Keywords: Mannheim curves, Lorentzian space, curvature, torsion.

INTRODUCTION

As is well-known, a surface is said to be "ruled" if it is generated by moving a straight line continuously in Euclidean space (O'Neill, 1997). Ruled surfaces are one of the simplest objects in geometric modeling. One important fact about ruled surfaces is that they can be generated by straight lines. A practical application of these type of surfaces is that they are used in civil engineering and physics (Guan et al., 1997).

Since building materials, such as wood are straight, they can be considered as straight lines. The result is that if engineers are planning to construct something with curvature, they can use a ruled surface since all the lines are straight (Orbay et al., 2009). In the differential geometry of a regular curve in the Euclidean 3-space IE^3, it is well-known that one of the important problem is the characterization of a regular curve. The curvature functions k_1 and k_2 of a reguler curve play an important role to determine the shape and size of the curve (Kuhnel, 1999; Do Carmo, 1976). For example, If $k_1 = k_2 = 0$, the curve is geodesic. If $k_1 \neq 0\ (constant)$ and $k_2 = 0$, then the curve is a circle with radius $1/k_1$. If $k_1 \neq 0\ (constant)$ and $k_2 \neq 0\ (constant)$, then the curve is a helix in the space.

Another way to the classification and characterization of curves is the relationship between the Frenet vectors of the curves. For example Saint Venant proposed the question whether upon the surfaces generated by the principal normal of a curve, a second curve can exist which has for its principal normal the principal normal of the given curve. This question was answered by Bertrand in 1850; he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients exists between the first and second curvatures of the given original curve. The pairs of curves of this kind have been called conjugate Bertrand curves, or more commonly Bertrand curves. There are many works related with Bertrand curves in the Euclidean space and Minkowski space. Another kind of associated curves are called Mannheim curve and Mannheim partner curve. If there exists a corresponding relationship between the space curves α and β such that, at the corresponding points of the curves, principal normal lines of α coincides with the binormal lines of β, then α is called a Mannheim curve, and β Mannheim partner curve of α.

In recent studies, Liu and Wang (2007, 2008) are curious about the Mannheim curves in both Euclidean and Minkowski 3-space and they obtained the necessary and sufficient conditions between the curvature and the torsion for a curve to be the Mannheim partner curves. Meanwhile, the detailed discussion concerned with the Mannheim curves can be found in literature (Wang and

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Liu, 2007; Liu and Wang, 2008; Orbay and Kasap, 2009; Özkaldi et al., 2009; Azak, 2009) and references therein.

In this paper, we study the timelike-spacelike Mannheim partner curves in Lorentzian space IR_1^3.

PRELIMINARY

The Minkowski 3-space IR_1^3 is the real vector space IR^3 provided with the standart flat metric given by:

$$\langle \, , \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a standard rectangular coordinate system of IR_1^3. An arbitrary vector $v = (v_1, v_2, v_3)$ in IR_1^3 can have one of three Lorentzian causal characters; it can be spacelike if $\langle \vec{v}, \vec{v} \rangle > 0$ or $v = 0$, timelike if $\langle \vec{v}, \vec{v} \rangle < 0$ and null (lightlike) if $\langle \vec{v}, \vec{v} \rangle = 0$ and $v \neq 0$.

Similarly, an arbitrary curve $\alpha = \alpha(s)$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha(s)$ are spacelike, timelike or null (lightlike), respectively. We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. The norm of the vector $v = (v_1, v_2, v_3) \in IR_1^3$ is given by:

$$\|\vec{v}\| = \sqrt{|\vec{v}|}$$

For any vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in IR_1^3 in the meaning of Lorentz vector product of x and y is defined by:

$$\vec{x} \wedge \vec{y} = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, -\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right)$$

The Lorentzian sphere and hyperbolic sphere of radius r and center 0 in IR_1^3 are given by:

$$S_1^2 = \left\{ \vec{x} = (x_1, x_2, x_3) \in IR_1^3 : \langle \vec{x}, \vec{x} \rangle = r^2 \right\}$$

and

$$H_0^2 = \left\{ \vec{x} = (x_1, x_2, x_3) \in IR_1^3 : \langle \vec{x}, \vec{x} \rangle = -r^2 \right\}$$, respectively

Definition 1

Hyperbolic angle

Let a and b be timelike vectors in IR_1^3. Then, the angle between a and b is defined by $\langle \vec{a}, \vec{b} \rangle = -\|\vec{a}\| \|\vec{b}\| \cosh\theta$. The number θ is called the hyperbolic angle.

Central angle

Let a and b be spacelike vectors in IR_1^3 that span a timelike vector subspace. Then, the angle between a and b is defined by $\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \|\vec{b}\| \cosh\theta$. The number θ is called the central angle.

Spacelike angle

Let a and b be spacelike vectors in IR_1^3 that span a spacelike vector subspace. Then, the angle between a and b is defined by $\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \|\vec{b}\| \cos\theta$. The number θ is called the spacelike angle.

Lorentzian timelike angle

Let a be a spacelike vector and b be a timelike vector in IR_1^3. Then, the angle between a and b is defined by $\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \|\vec{b}\| \sinh\theta$. The number θ is called the Lorentzian timelike angle.

Let $\{t(s), n(s), b(s)\}$ be the moving Frenet frame along the curve $\alpha(s)$. Then $t(s)$, $n(s)$ and $b(s)$ are tangent, the principal normal and the binormal vector of the curve $\alpha(s)$, respectively. Depending on the casual character of the curve α, we have the following dual Frenet-Serret formulas. If α is a timelike curve;

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \qquad (1)$$

where

$$\langle t, t \rangle = -1, \langle n, n \rangle = \langle b, b \rangle = 1, \langle t, n \rangle = \langle n, b \rangle = \langle t, b \rangle = 0.$$

We denote by $\{v_1(s), v_2(s), v_3(s)\}$ the moving Frenet frame along the curve $\beta(s)$. Then, $v_1(s)$, $v_2(s)$ and $v_3(s)$ are tangent, the principal normal and the binormal vector of the curve $\beta(s)$, respectively. Depending on the casual character of the curve β, we have the following Frenet-Serret formulas. If β is a spacelike curve with a timelike binormal v_3;

$$\begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} 0 & p & 0 \\ -p & 0 & q \\ 0 & q & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{2}$$

where

$$\langle v_3, v_3\rangle = -1, \langle v_1, v_1\rangle = \langle v_2, v_2\rangle = 1, \langle v_1, v_2\rangle = \langle v_2, v_3\rangle = \langle v_1, v_3\rangle = 0.$$

If the curves are unit speed curve, then curvature and torsion are calculated by:

$$\kappa_1 = \|t'\|,$$
$$k_2 = \langle n', b\rangle, \tag{3}$$
$$p = \|v_1'\|,$$
$$q = \langle v_2', v_3\rangle.$$

If the curves are not unit speed curve, then curvature and torsion are calculated by:

$$\kappa_1 = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3},$$
$$\kappa_2 = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}, \tag{4}$$

$$p = \frac{\|\dot\beta \wedge \ddot\beta\|}{\|\dot\beta\|^3},$$
$$q = \frac{\det(\beta', \beta'', \beta''')}{\|\beta' \wedge \beta''\|^2}.$$

TIMELIKE-SPACELIKE MANNHEIM PARTNER CURVE IN IR_1^3

Here, we define timelike-spacelike Mannheim partner curves in IR_1^3 and we give some characterization for timelike-spacelike Mannheim partner curves in the same space. Using these relationships, we will comment on Shell's and Mannheim's theorems again.

Definition 2

Let $\alpha : I \to IR_1^3$ be a timelike curve and $\beta : I \to IR_1^3$ be spacelike with timelike binormal. If there exists a corresponding relationship between the timelike curve α and the spacelike curve with dual timelike binormal β such that, at the corresponding points of the curves, the binormal lines of α coincides with the principal normal lines of β, then α is called a timelike Mannheim curve, and β is called a Mannheim partner curve of α. The pair $\{\alpha, \beta\}$ is said to be timelike-spacelike Mannheim pair. Let $\{t, n, b\}$ be the Frenet frame field along $\alpha = \alpha(s)$ and let $\{v_1, v_2, v_3\}$ be the Frenet frame field along $\beta = \beta(s)$. On the other way θ is angle between t and v_1, there is a following equation between the Frenet vectors and their derivative;

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \\ \cosh\theta & \sinh\theta & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}. \tag{5}$$

Theorem 1

The distance between corresponding points of the timelike-spacelike Mannheim partner curves in IR_1^3 is constant.

Proof

From the definition of spacelike Mannheim curve, we can write:

$$\tilde\beta(s^*) = \tilde\alpha(s) + \lambda(s) B(s). \tag{6}$$

By taking the derivate of this equation with respect to s and applying the Frenet formulas, we get:

$$v_1 \frac{ds^*}{ds} = t - \lambda k_2 n + \lambda' b \tag{7}$$

where the superscript $(')$ denotes the derivate with respect to the arc length parameters of the dual curve

$\alpha(s)$. Since the vectors b and v_2 are linear, we get :

$$\left\langle v_1 \frac{ds^*}{ds}, b \right\rangle = \langle t, b\rangle - \lambda k_2 \langle n, b\rangle + \lambda \langle b, b\rangle, \lambda' = 0.$$

Then, we get $\lambda = c$. On the other hand, from the definition of distance function between $\alpha(s)$ and $\beta(s)$ we can write:

$$d(\alpha(s), \beta(s)) = \left\| \lambda(s) B(s) \right\| = \lambda .$$

This completes the proof.

Theorem 2

For a timelike-spacelike curve α in IR_1^3, there is a spacelike curve β so that $\{\alpha, \beta\}$ is a spacelike Mannheim pair.

Proof

Since the vectors v_2 and b are linearly dependent, the Equation 6 can be written as:

$$\alpha = \beta - \lambda v_2 . \tag{8}$$

Since λ is a nonzero constant, there is a timelike curve β for all values of λ.

Now, we can give the following theorem related to curvature and torsion of the timelike-spacelike Mannheim partner curves.

Theorem 3

Let $\{\alpha, \beta\}$ be a timelike-spacelike Mannheim pair in IR_1^3. If k_2 is dual torsion of α and p is dual curvature and q is dual torsion of β, then:

$$k_2 = -\frac{p}{\lambda q}. \tag{9}$$

Proof

By taking the derivate of Equation 7 with respect to s and applying the Frenet formulas, we obtain:

$$v_1 \frac{ds^*}{ds} = t - \lambda k_2 n. \tag{10}$$

Let θ be dual angle between the dual tangent vectors t and v_1, we can write:

$$\begin{aligned} v_1 &= \sinh\theta\, t + \cosh\theta\, n \\ v_3 &= \cosh\theta\, t + \sinh\theta\, n. \end{aligned} \tag{11}$$

From Equations 10 and 11 , we get:

$$\frac{ds^*}{ds} = \frac{1}{\sinh\theta}, \quad -\lambda k_2 = \cosh\theta \frac{ds^*}{ds} . \tag{12}$$

By taking the derivate of Equation 8 with respect to s and applying the Frenet formulas, we obtain:

$$k_2 = (1 + \lambda p) v_1 \frac{ds^*}{ds} - \lambda q v_3 \frac{ds^*}{ds} . \tag{13}$$

From Equation 11, we can write:

$$\begin{aligned} t &= -\sinh\theta\, v_1 + \cosh\theta\, v_3 \\ n &= \cosh\theta\, v_1 - \sinh\theta\, v_3 , \end{aligned} \tag{14}$$

where θ is the angle between t and v_1 at the corresponding points of the curves of α and β. By taking into consideration Equations 13 and 14, we get:

$$\sinh\theta = -(1 + \lambda p)\frac{ds^*}{ds}, \quad \cosh\theta = -\lambda q \frac{ds^*}{ds} . \tag{15}$$

Substituting $\frac{ds^*}{ds}$ into Equation 15 , we get:

$$\sinh^2\theta = -(1 + \lambda p), \cosh^2\theta = \lambda^2 k_2 q. \tag{16}$$

From the Equation 16, we can write:

$$k_2 = -\frac{p}{\lambda q}.$$

Corollary 1

Let $\{\alpha, \beta\}$ be a timelike-spacelike Mannheim pair in IR_1^3. Then, the product of torsions k_2 and q at the corresponding points of the spacelike Mannheim partner curves are not constant.

Namely, Schell's theorem is invalid for the timelike-spacelike Mannheim curves. By considering Theorem 3, we can give the following results.

Corollary 2

Let $\{\alpha, \beta\}$ be a timelike-spacelike Mannheim pair in IR_1^3. Then, torsions k_2 and q has a negative sign.

Theorem 4

Let $\{\alpha, \beta\}$ be a timelike-spacelike Mannheim pair in IR_1^3. Between the curvature and the torsion of the spacelike curve β, there is the relationship:

$$\mu q - \lambda p = 1 \tag{17}$$

where $\propto$ and λ are nonzero dual numbers.

Proof

From Equation 15 , we obtain:

$$\frac{\sinh\theta}{1+\lambda p} = \frac{\cosh\theta}{\lambda q} \tag{18}$$

Arranging this equation, we get:

$$\tanh\theta = \frac{1+\lambda p}{\lambda q}$$

and if we choose $\mu = \lambda \tanh\theta$ for brevity, we will see that: $\mu q - \lambda p = 1$.

Theorem 5

Let $\{\alpha, \beta\}$ be a timelike-spacelike Mannheim pair in IR_1^3. The following equations are for the curvatures and the torsions of the curves α and β

1. $k_1 = -\dfrac{d\theta}{ds}$,

2. $k_2 = p\cosh\theta\dfrac{ds^*}{ds} - q\sinh\theta\dfrac{ds^*}{ds}$,

3. $p = k_2\cosh\theta\dfrac{ds}{ds^*}$,

4. $q = k_2\sinh\theta\dfrac{ds}{ds^*}$.

Proof

1. By considering Equation 11, we can easily show that $\langle t, v_1\rangle = \sinh\theta$. Differentiating this equality with respect to s by considering Equation 1 , we have:

$$\langle t', v_1\rangle + \langle t, v_1'\rangle = -\sinh\theta\frac{d\theta}{ds},$$

From Equations 1 and 2, we can write:

$$\langle k n, v_1\rangle + \langle t, p v_2\rangle\frac{ds^*}{ds} = -\sinh\theta\frac{d\theta}{ds}$$

From Equation 14, we get:

$$k_1 = -\frac{d\theta}{ds}.$$

2. By considering Equation 11, we can easily show that $\langle n, v_2\rangle = 0$. Differentiating this equality with respect to s and by considering Equation 1, we have:

$$\langle n', v_2\rangle + \langle n, v_2'\rangle\frac{ds^*}{ds} = 0,$$

From Equations 1 and 2, we can write:

$$\langle k_1 t + k_2 b, v_2\rangle + \langle\cosh\theta\, v_1 - \sinh\theta\, v_3, (-p v_1 + q v_3)\rangle\frac{ds^*}{ds} = 0,$$

From Equations 14, we get:

$$k_2 = p\cosh\theta\frac{ds^*}{ds} - q\sinh\theta\frac{ds^*}{ds}.$$

3. By considering Equation 14, we can easily show that $\langle b, v_1\rangle = 0$. Differentiating this equality with respect to s and by considering Equation 1 , we have:

$$\left\langle b', v_1 \right\rangle + \left\langle b, v_1' \frac{ds^*}{ds} \right\rangle = 0,$$

From Equations 1, 2 and 14, we can write:

$$\left\langle -k_2 \left(\cosh\theta\, v_1 - \sinh\theta\, v_3 \right), v_1 \right\rangle + \left\langle b, p v_2 \frac{ds^*}{ds} \right\rangle = 0,$$

$$p = k_2 \cosh\theta \frac{ds}{ds^*}.$$

4. By considering Equation 11, we can easily show that $\langle b, v_3 \rangle = 0$. Differentiating this equality with respect to s and by considering Equation 1, we have:

$$\left\langle b', v_3 \right\rangle + \left\langle b, v_3' \frac{ds^*}{ds} \right\rangle = 0.$$

From Equations 1, 2 and 14, we can write:

$$\left\langle -k_2 \left(\cosh\theta\, v_1 - \sinh\theta\, v_3 \right), v_3 \right\rangle + \left\langle b, q v_2 \frac{ds^*}{ds} \right\rangle = 0,$$

$$q = k_2 \sinh\theta \frac{ds}{ds^*}.$$

By considering statements 3 and 4 of Theorem 5, we can give the following results.

Corollary 3

Let $\{\alpha, \beta\}$ be a timelike-spacelike Mannheim pair in IR_1^3. Then, there exist the following relation between curvature and torsion of β and torsion of α;

$$p^2 - q^2 = k_2^2 \left(\frac{ds}{ds^*}\right)^2. \quad (19)$$

Theorem 6

A timelike space curve in IR_1^3 is a timelike-spacelike Mannheim curve, if and only if its curvature p and torsion q satisfy the formula:

$$\lambda\left(q^2 - p^2\right) = p \quad (20)$$

where λ is never a pure dual constant.

Proof

By taking the derivate of the statement $\alpha = \beta - \lambda V_2$ with respect to s and applying the Frenet formulas we obtain:

$$t \frac{ds^*}{ds} = v_1 + \lambda \left(p v_1 - q v_3 \right),$$

$$k_1 n \left(\frac{ds}{ds^*}\right)^2 + t \frac{d^2 s}{ds^{*2}} = p v_2 + \lambda \left(p' v_1 - q' v_3 + \left(p^2 - q^2\right) v_2 \right).$$

Taking the inner product, which is the last equation with b, we get:

$$\lambda\left(q^2 - p^2\right) = p.$$

Theorem 7

Let $\{\alpha, \beta\}$ be a timelike-spacelike Mannheim partner curves in IR_1^3. Moreover, the points $\alpha(s)$ and $\beta(s)$ be two corresponding points of $\{\alpha, \beta\}$ and M and M^* be the curvature centers at these points, respectively. Then, the ratio:

$$\frac{\left\| \beta(s) M \right\|}{\left\| \alpha(s) M \right\|} : \frac{\left\| \beta(s) M^* \right\|}{\left\| \alpha(s) M^* \right\|} = \left(1 + k_1 p\right)\left(1 + \lambda p\right) \neq \text{constant}. \quad (21)$$

Proof

A circle that lies in the osculating plane of the point $\alpha(s)$ on the timelike curve α and that has the centre $M = \alpha(s) + \frac{1}{k_1} n$ lying on the principal normal n of the point $\alpha(s)$ and the radius $\frac{1}{k_1}$ far from $\alpha(s)$, is called osculating circle of the curve α in the point $\alpha(s)$. Similar definition can be given for curve β too.

Then, we can write:

$$\left\| \alpha(s) M \right\| = \left\| \frac{1}{k_1} n \right\| = \frac{1}{k_1},$$

$$\left\|\alpha(s)M^*\right\| = \left\|\lambda b + \frac{1}{p} v_2\right\| = \frac{1}{p} + \lambda ,$$

$$\left\|\beta(s)M^*\right\| = \left\|\frac{1}{p} v_2\right\| = \frac{1}{p} ,$$

$$\left\|\beta(s)M\right\| = \left\|\lambda v_3 + \frac{1}{k_1} n\right\| = \frac{1}{k_1} + \lambda$$

Therefore, we obtain:

$$\frac{\left\|\beta(s)M\right\|}{\left\|\alpha(s)M\right\|} : \frac{\left\|\beta(s)M^*\right\|}{\left\|\alpha(s)M^*\right\|} = (1+\lambda p)\sqrt{1-\lambda^2 k_1^2} \neq \text{CONSTANT}.$$

Thus, we can give the following.

Corollary 4

Mannheim's theorem is invalid for the timelike-spacelike Mannheim partner curve $\{\alpha, \beta\}$ in IR_1^3.

REFERENCES

Azak AZ (2009). On Timelike Mannheim Partner Curves in L_3 Sakarya University Faculty of Arts and Science, The Journal of Arts and Science, (11): 35-45.

Do Carmo MP (1976). Differential Geometry of Curves and Surfaces, Pearson Education. New York: Academic Press.

Guan Z, Guan, Ling J, Ping X, Rongxi T (1997). Study and Application of Physics-Based Deformable Curves and Surfaces. Computers and Graphics, 21: 305-313.

Kahraman T, Önder M, Kazaz M, Uğurlu HH (2011). Some Characterizations of Mannheim Partner Curves in the Minkowski 3-E_1^3 space. Proceedings of the Estonian Academy of Sciences, 60, 4, 210–220.

Kuhnel W (1999). Differential Geometry: Curves-Surfaces-Manifolds. Braunschweig, Wiesbaden.

Liu H, Wang F (2007). Mannheim Partner Curves in 3-Space. Procedings of The Eleventh Int. Workshop on Diff. Geom., 11: 25-31

Liu H, Wang F (2008). Mannheim Partner Curves in 3-Space, J. Geom., 88(1-2): 120-126.^{3}

Orbay K, Kasap E (2009). On Mannheim Partner Curves in E . Int. J.

Orbay K, Kasap E, Aydemir İ (2009). Mannheim Offsets of Ruled Surfaces. Mathematical Problems in Engineering. Article Number: 160917.

O'Neill B (1997). Elemantary Differential Geometry, 2nd ed. Academic Press, New York.

Özkaldi S, İlarslan K, Yaylı Y (2009). On Mannheim Partner Curves In Dual Space. Analele Stiintifice ale Universitatii Ovidius Constanta. Seria Matematica. XVII, fasc. 2.